

Ex 3.2

Basic

1. If V is the volume of a cube and x is the length of an edge.

Express $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$. What is $\frac{dV}{dt}$ when x is 5 and $\frac{dx}{dt} = 2$?

2. If V is the volume of a sphere and r is the radius.

Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$. What is $\frac{dV}{dt}$ when $r=4$ and $\frac{dr}{dt} = 2$?

3. If V is the volume of a cone and h is the height and r is the radius.

Express $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is $\frac{1}{2}$ of h . What is $\frac{dV}{dt}$ when $\frac{dh}{dt} = 5$ and $h=2$?

Cube

4. If the side of a cube increases at a rate of 1 cm/s, find the rate of increase of the total surface area of the cube when the side of the cube is 20 cm.

5. The side of a cube increases at 1 cm/s. How fast is the diagonal of the cube changing when the side is 1 cm?

Circle

6. A pebble is thrown into water and causes circular ripple to spread outward at a rate of 2 m/s.

Find the rate of change of area in terms of π when a) 3 seconds after the pebble falls into the water.

b) the area of the ripple is $9\pi \text{ m}^2 / \text{s}$.

7. Radius of a circle varies as time t in m/s according to the following rule $r = t^3 + 2t$. Find the rate of change of the area at $t = 2$.

Rectangles

8. The side of a square is increasing at a rate of 1 cm/s. Find the rate of change of area when the area is 100 cm^2 .

9. The width of a rectangle is decreasing at a rate of 1 cm/s and the length is increasing at a rate of 2 cm/s. Find the rate of change of the area at the moment when both the width and the length are 10 cm.

Rectangular tank

10. The base of a rectangular tank is 3 m by 4 m and is 10 m high. Water is added at a rate of $8 \text{ m}^3/\text{min}$. Find the rate of change of water level when the water is 5 m deep.

11. The dimensions $W \times L \times H$ of a rectangular box is increasing at rates of 1 cm/s, 2 cm/s, and 3 cm/s respectively. Find the rate of change of the volume at the instant when $W = 1 \text{ cm}$, $L = 2 \text{ cm}$, and $H = 3 \text{ cm}$.

Cylinder

12. Water is added to a cylindrical tank of radius 5 m and height 10 m at a rate of 100 L/min. Find the rate of change of the water level when the water is 6 m deep. ($1 \text{ L} = 1000 \text{ cm}^3$)

13. The radius of a cylinder is decreasing at a rate of 1 cm/min. The height remains the same as 20 cm. How fast is the volume changing when the radius is 12 cm?

Sphere

14. A sphere is growing at a rate of $1 \text{ cm}^3/\text{s}$. Find the rate at which the radius is changing when the radius is 2 cm.

15. A balloon is rising at a rate of 200 m/min. For every 500 m increase in height the air pressure causes the radius to increase 2 cm. Find the rate of increase of the volume when the radius is 10 m.

16. A spherical balloon is inflated at a rate of $4\pi \text{ cm}^3/\text{s}$. At the moment the radius is 5 cm, find the rate of change of
 a) the radius b) the surface area

Cone

17. Sand is poured by a conveyor belt at a rate of $2 \text{ m}^3/\text{min}$ onto the ground which forms a conical pile in which the radius is always twice the height. How fast is the height rising when the height is 4 m?
18. The dimensions of a conical tank is of radius 3 m and height 6 m. Water is added to it at a rate of $\pi \text{ m}^3/\text{min}$. Find the rate of change of the water level when the height is 3 m.

Ladder

19. A ladder is 10 m long and leans against a vertical wall. If the foot of the ladder is moving away at a rate of 2 m/s and the foot is 6 m away from the wall, find the rate of change
 a) of the top of the ladder that is moving down b) of the slope of the ladder
20. An extension ladder with the top resting on a vertical wall is being extended at a rate of 10 m/min. The base is 5 m from the wall. Find the rate of sliding up of the top of the ladder when the ladder is 13 m long.
21. A ladder is resting on a vertical wall. The foot of the ladder is sliding away at a rate of 4 m/s and is 12 m from the wall. At this moment the top of the ladder is moving down at a rate of 3 m/s. Find the length of the ladder.
22. A wall is inclined 60 degree to the ground. A ladder $4\sqrt{3}$ m is resting on the wall. The foot of the ladder is moving away at a rate of 2 m/s. How fast is the top sliding down at the instant when the foot of the ladder is 4 m away from the foot of the wall?



Intersection

23. Two objects are moving away from the same location at the same time. Object A is moving due east at a rate of 3 cm/s. Object B is moving due north at a rate of 4 cm/s. Find the rate of change of distance between the two objects 2 seconds later.
24. Ship A is 25 km south of ship B and is moving due north at a rate of 5 km/h. Ship B is moving due east at a rate of 10 km/h. Find the rate of change of the distance between the two ships 2 hours later.
25. Object A is 15 m east of object B and is moving due north at a rate of 2 m/s while object B is moving due south at a rate of 3 m/s. Find the rate of change of the distance between the two objects 4 seconds later.
26. At noon ship A leaves a location and is moving due east at the rate of 10 km/h. One hour later another ship B starts to leave the same location and is moving at a rate of 15 km/h. Ship B moves in the direction N30 E. Find the rate of change of distance between the two ships at 3:00 p.m.
27. A small snake (height negligible) moves along a straight path at a speed of 2 m/s passes a tree (10 m tall) 4 m away. How fast is the distance between the top of the tree and the snake changing 2 seconds later?
28. A dog runs across a bridge at a rate of 4 m/s. The bridge is 5 m above the water. A fish swims right underneath the dog at a rate of 3 m/s in a direction perpendicular to the path of the dog. Find the rate of change of the distance between the two animals 1 second later.
29. At noon ship A is 10 km west of ship B and is moving due east at a speed of 10 km/h. Ship B is moving due south at a rate of 5 km/h. When is the distance between them stop changing?

Trough

30. A water trough is 10 m long and has a cross section of an equilateral triangle. Water is added at a rate of $2 \text{ m}^3/\text{min}$. Find the rate of change of the water level when the water is $\sqrt{3}$ m deep.

31. A trough is 10 m long and its ends are isosceles trapezoids with base 2 m, top 3 m, and height 4 m. Water is added at a rate of $5 \text{ m}^3/\text{min}$. Find the rate of change of the water level when the water is 2 m deep.

Shadow

32. A man 2 m tall is walking away from a lamppost which is 6 m tall at a rate of 2 m/s. Find the rate of change of
a) the tip of his shadow b) the length of his shadow
33. A spot light on the ground is shining on a vertical building which is 20 m from the spot light. A man 2 m tall is walking away from the light and is walking directly towards the building at a rate of 4 m/s . Find the rate of change of the length of his shadow on the building when he is 5 m from the light.

Kite

34. A boy is flying a kite. The string of the kite is being paid out at a rate of 10 m/min . The kite is moving horizontally and is at a height of 150 m above the ground. Find the rate at which the kite is moving when it is 250 m away from the boy.
35. A boat which is 3 m below the dock is being pulled in by a cable. When the boat is 4 m away horizontally from the dock it is approaching the dock horizontally at a rate of 5 m/s . How fast is the cable being pulled in?
36. A fisherman is on a bridge 3 m above the water. He reels in his line at a rate of 1 m/s thinking that he will get a big fish. But actually it is a piece of wood. (So it remains on the surface of the water). How fast is the piece of wood approaching the bridge when it is 4 m from the foot of the bridge?
37. A pulley is 15 m above the ground and a box is located right under the pulley. A rope 30 m long passes over the pulley and fastened to the box. The other end of the rope moves horizontally away at a rate of 5 m/s . How fast is the box rising when it is 10 m high?

Moving along curves

38. Point $P(x,y)$ moves along the parabola $y=x^2$ from left to right at a rate of 5 unit/s . Find dy/dt at the moment it passes the point $(4, 16)$.
39. A point moves along the curve $y=x^3$ from left to right with at a rate of 2 unit/s . Find the rate of change of the slope of the curve at $x = 4$.
40. Area bounded by $y=x^3$, the x-axis and the line $x = a$ ($a > 0$) is given by $A = \frac{a^4}{4}$. If the line $x = a$ is moving to the right at a rate of 2 units/s . Find the rate of change of area at the moment $x = 2$.

Miscellaneous

41. There is a spherical marble of radius 1 cm in the middle of a hemispherical bowl which is of radius 2 cm. Water is added at a rate of $\pi \text{ cm}^3/\text{s}$. Find the rate of change of the water level when the water is 1 cm deep. (Note: You may skip this problem if you do not know the formula $dV/dt = (A)(dh/dt)$.)
42. The intensity of illumination I of a point r unit from a light source is inversely proportional to the square of the distance r unit from the light source. i.e. $I = \frac{k}{r^2}$. Given $I = 100$ when $r = 1$.
An object is moving away from the light source at a rate of 4 unit/s . Find the rate of change of illumination when it is 2 unit from the light source.
43. A plane 2 km high is flying at a rate of 120 km/h due west sees an oncoming car. The distance between the plane and the car is 4 km and is decreasing at a rate of 160 km/h . Find the speed of the car at this moment.

Chapter 7 Integration and Area

Something U Need to Know

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int kf(x) dx = k \int f(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad \text{if } n \neq -1$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \csc^2 x dx = -\cot x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \csc x \cot x dx = -\csc x + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int u dv = uv - \int v du$$

Integration by Summation

Ex 7.1

Using summation to evaluate the following definite integrals

1. $\int_0^1 c dx$

2. $\int_0^1 x dx$

3. $\int_0^1 x^2 dx$

4. $\int_0^1 x^3 dx$

5. $\int_0^1 (x^2 + 2x) dx$

6. $\int_2^4 x^2 dx$

7. $\int_1^4 2x^3 dx$

Integration by Basic Formula

Ex 7.2

Find the general antiderivatives, given y' equal to:

1. 3

2. $5x$

3. $8x^2$

4. x^7

5. x^7

6. x^e

7. $7\sqrt{x}$

8. $\frac{1}{\sqrt{x}}$

9. $\sqrt[3]{x^4}$

10. $\sqrt{xx^3}$

11. $(3x-8)(x+2)$

12. $\frac{x+1}{\sqrt{x}}$

13. $\frac{\sqrt[3]{x^2} - \sqrt[3]{x}}{\sqrt{x}}$

14. $\frac{x-1}{\sqrt[3]{x}-1}$

15. $x(x+1)(x+2)$

16. $a+bx+cx^2$

17. $\frac{(2x+1)(x-3)}{x^4}$

18. $\left(\sqrt{x} + \frac{1}{x}\right)^2$

19. $(\sqrt{1-x^2} - \sqrt{x})^2 + (\sqrt{1-x^2} + \sqrt{x})^2$

20. $\frac{1}{\cos^2 x}$

21. $3\sin x - 4\cos x$

22. $2\sin \frac{x}{2} \cos \frac{x}{2}$

23. $(\sec x + \tan x) \tan x$

24. $\frac{3}{\sin^2 x} + \frac{4}{\cos^2 x}$

27. $(e^x + 1)^2$

30. $\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2$

33. $(\tan 5x + \sec 5x)^2$

36. $e^{\ln \sin(x^{(2 \sin^2 x - \cos^2 x)})}$

25. $\frac{1+3\cos^3 x}{\cos^2 x}$

28. $\frac{1}{1-\cos x}$

31. $(\csc^2 x - \cot^2 x)^{10}$

34. $(e^{2x} - e^{-2x})^2$

26. $\tan^2 x$

29. $\frac{\cos 2x}{\sin^2 2x}$

32. $e^{7x(\sec^2 x - \tan^2 x)}$

35. $\frac{\sin kx}{\cos^2 kx}$

Geometric Applications and Motion

unless otherwise stated, take the gravitational acceleration as 10 m/s^2 .

Ex 7.3 (s stands for distance, v for velocity, a for acceleration and t for time.)

- Given $y' = 8x - 6x^2$, and $y = 40$ when $x = 1$. Find y .
- $y'' = 6x - 4$, and y passes $(0, 3)$ and $(1, -4)$. Find y .
- Given $y'' = kx + 6$, and y passes $(0, 0)$ and has a slope of 3 at point $(1, 5)$. Find y .
- Given $a = 3t^2 + 2t$, find v given $v = 2$ when $t = 0$.
- If $v = -t + 2$. Find the distance traveled from $t = 0$ to $t = 2$.
- A particle starts from rest at point 0 with acceleration $a = 10 - 2t$. When and where will it come to rest again?
- A man travels from rest from place X to place Y with velocity $v = 6t - 3t^2 \text{ km/h}$ without stopping on his travel.
 - Find the traveling time.
 - Find the distance between X and Y
 - Find the maximum velocity during his travel.
- A pebble is released from a height of 20 m above the ground. When will it reach the ground and what is the velocity?
- A pebble is thrown downward with a velocity of 20 m/s from the top of a building and reach the ground in 2 seconds. How high is the building?
- A man can jump 3 m (super!) on the earth whose gravitation acceleration is 10 m/s^2 . How high can he jump on the moon with gravitational acceleration 1.6 m/s^2 ?

Method of Substitution

Ex 7.4 Evaluate the following integrals

1. $\int \sqrt{2x+1} dx$

2. $\int x^2 \sqrt{1-x} dx$

3. $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx$

4. $\int \frac{4x}{\sqrt{1-8x^2}} dx$

5. $\int x(x^2+4)^{10} dx$

6. $\int \frac{2x^2}{(3x^3+2)^4} dx$

7. $\int \sqrt{x} \sqrt{4+x} \sqrt{x} dx$

8. $\int (x+1)(x-3)^{10} dx$

9. $\int (2x+10)(x^2+10x+8)^5 dx$

10. $\int \frac{x+4}{\sqrt[3]{x^2+8x+1}} dx$

11. $\int \frac{x^2}{\sqrt{3+x}} dx$

12. $\int \frac{x^2}{(x+1)^7} dx$

13. $\int \frac{\left(4 - \frac{3}{x}\right)^4}{x^2} dx$

14. $\int \frac{1}{x^3} \left(4 - \frac{1}{x^2}\right)^{\frac{1}{5}} dx$

15. $\int \frac{x^2+4}{x^3+12x+10} dx$

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|---|--|--|
| 16. $\int \frac{2x}{(2x^2+1)^3} dx$ | 17. $\int \frac{1}{x^2} \sqrt{1-\frac{1}{x}} dx$ | 18. $\int (4x^2+4x+1)^{10} dx$ |
| 19. $\int \frac{dx}{x^2+2x+2}$ | 20. $\int \frac{\ln x}{x} dx$ | 21. $\int \frac{\ln \sqrt{x}}{x} dx$ |
| 22. $\int \frac{1}{x \ln x} dx$ | 23. $\int (1+\sqrt{x})^{100} dx$ | 24. $\int \frac{e^x}{1-e^x} dx$ |
| 25. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ | 26. $\int x^2 e^{x^3} dx$ | 27. $\int \frac{e^x}{e^{2x}+1} dx$ |
| 28. $\int e^{x+e^x} dx$ | 29. $\int \frac{dx}{\csc x - \cot x}$ | 30. $\int \frac{\sin x}{\sqrt{\cos x}} dx$ |
| 31. $\int \cos^3 2x dx$ | 32. $\int \frac{\cos^3 x}{\sin^2 x} dx$ | 33. $\int \tan^5 x \sec^2 x dx$ |
| 34. $\int \frac{4 \cos x}{(4+\sin x)} dx$ | 35. $\int x \sin(3-x) dx$ | 36. $\int \frac{x^2 \csc^2(2x^3)}{\sqrt{1-\cot(2x^3)}} dx$ |
| 37. $\int x^2 \sec^2(x^3) dx$ | 38. $\int \frac{\sin x}{(1+\cos x)} dx$ | 39. $\int \frac{4 \cos 2x}{\sin^5 2x} dx$ |

Integration by Parts

Ex 7.5 Evaluate the following integrals

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|-----------------------------|--------------------------|--------------------------|--------------------------------|
| 1. $\int \ln x dx$ | 2. $\int \tan^{-1} x dx$ | 3. $\int \ln(1+x) dx$ | 4. $\int x \cos x dx$ |
| 5. $\int x \sin kx dx$ | 6. $\int x e^x dx$ | 7. $\int x \ln x dx$ | 8. $\int \frac{\ln x}{x^3} dx$ |
| 9. $\int x \tan^{-1} x dx$ | 10. $\int x e^{kx} dx$ | 11. $\int e^x \sin x dx$ | 12. $\int e^{\sqrt{x}} dx$ |
| 13. $\int \sin \sqrt{x} dx$ | 14. $\int x^2 \ln x dx$ | 15. $\int x^2 \sin x dx$ | 16. $\int x^2 e^x dx$ |

Integration of Rational Functions

Ex 7.6 Evaluate the following integrals

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|---------------------------------|---------------------------------------|------------------------------------|----------------------------------|
| 1. $\int \frac{x+7}{x} dx$ | 2. $\int \frac{x+1}{x+2} dx$ | 3. $\int \frac{x^2+1}{x-1} dx$ | 4. $\int \frac{x^2+1}{x^2-1} dx$ |
| 5. $\int \frac{dx}{5x+6}$ | 6. $\int \frac{dx}{(5x+6)^2}$ | 7. $\int \frac{5x+8}{(x+7)^2} dx$ | 8. $\int \frac{x+4}{x^2-4} dx$ |
| 9. $\int \frac{x}{x^2+3x+2} dx$ | 10. $\int \frac{x^2-2x}{x^2-4x+3} dx$ | 11. $\int \frac{3x+4}{(x+1)^3} dx$ | 12. $\int \frac{1}{x^2(x+1)} dx$ |

Integration of Trigonometric Functions

Ex 7.7 Evaluate the following integrals

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|-----------------------|------------------------|------------------------|------------------------|
| 1. $\int \sin kx dx$ | 2. $\int \cos kx dx$ | 3. $\int \tan kx dx$ | 4. $\int \cot kx dx$ |
| 5. $\int \sec kx dx$ | 6. $\int \csc kx dx$ | 7. $\int \sin^2 x dx$ | 8. $\int \cos^2 x dx$ |
| 9. $\int \tan^2 x dx$ | 10. $\int \cot^2 x dx$ | 11. $\int \sec^2 x dx$ | 12. $\int \csc^2 x dx$ |

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|---------------------------------|---------------------------------|---------------------------------|-------------------------------|
| 13. $\int \sin^3 x dx$ | 14. $\int \cos^3 x dx$ | 15. $\int \tan^3 x dx$ | 16. $\int \cot^3 x dx$ |
| 17. $\int \sec^3 x dx$ | 18. $\int \csc^3 x dx$ | 19. $\int \sin^4 x dx$ | 20. $\int \cos^4 x dx$ |
| 21. $\int \sin^5 x dx$ | 22. $\int \cos^5 x dx$ | 23. $\int \sec^4 x dx$ | 24. $\int \sec^6 x dx$ |
| 25. $\int \sin^3 x \cos^2 x dx$ | 26. $\int \sin^6 x \cos^3 x dx$ | 27. $\int \sin^2 x \cos^2 x dx$ | 28. $\int \tan x \sec^2 x dx$ |
| 29. $\int \tan^3 x \sec x dx$ | 30. $\int \sec^4 x dx$ | 31. $\int \sec^4 x \tan^4 x dx$ | |

Area

Ex 7.8 Find the area bounded by the following curves

1. Find the area bounded by the curve $y = -x^2 + x$ and the following straight lines.
 - a) x-axis
 - b) $x = -1, x = 0, x$ -axis
 - c) $x = -1, x = 1, x$ -axis
 - d) $x = -1, x = 2, x$ -axis.
2. The x-axis and the curve $y = x^2 - 4$.
3. $y = 5x - x^2, y = 0$ and $x = 4$.
4. $y = -x^2 + x + 6$ and the x-axis.
5. The curve $y = x^2 - 4x$ and the x-axis.
6. The curve $y = x^3 - 4x$ and the x-axis.
7. The curve $y = x^3 - 11x^2 + 38x - 40$ and x-axis from $x = 1$ to $x = 5$.
8. The curve $y = x^3 - 7x^2 + 14x^2 - 8$ from $x = 1$ to $x = 3$.
9. The curve $y = (x+1)(x-2)(x-4)$ and the x-axis.
10. The curve $x = 1 - y^2$ and the y-axis.
11. The curve $y^2 - x - 2 = 0$ and the y-axis.
12. The curves $y = x^2$ and $y^2 = x$
13. The curve $y = 4 - x^2$ and the line $y = 3x$.
14. The curve $y = x^2 + 1$ and the line $y = x + 7$.
15. The curves $y = \sin x$ and $y = \sin 2x$ from 0 to π .
16. The curve $y = x^3$ and the line $y = x$.
17. The curves $y = e^x$ and $y = e^{-x}$ and the x-axis from $x = -\ln 2$ to $\ln 2$.
18. The curves $y = (x-1)^2$ and $y = -x^2 + 5x - 2$.
19. The curves $y = x^2$ and $y = -\frac{1}{x^2}$ from $x = -2$ to $x = -1$.
20. The curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \pi$.
21. The line $y = x - 2$ and the curve $y^2 = x$.
22. The curve $y^2 - 6y + x = 0$ and $y = x$.
23. The curves $y^2 = -4x$ and $y^2 = -2(x-1)$.